Lecture 3. Separable Equations

Recall in Lecture 2, we solved questions like

$$\frac{dy}{dx} = f(x)$$

The idea is *integrating both sides*. Can we apply the same idea for the following question?

Example 1. Find solutions of the differential equation $\quad \dfrac{dy}{dx} = y \sin x.$ ① $= k(y) \cdot f(x)$

ANS: If $y \neq 0$, we can divide both sides by y, and multiply both sides by dx.

$$\frac{dy}{y} = \sin x \, dx$$

Integrate both sides, we have

$$\int \frac{dy}{y} = \int \sin x \, dx \Rightarrow \ln|y| = -\cos x + C_1$$

$$\Rightarrow \quad e^{\ln|y|} = e^{-\cos x + C_1} \Rightarrow |y| = e^{C_1} \cdot e^{-\cos x}$$

$$\Rightarrow \quad y = \underbrace{\pm e^{C_1}}_{\text{is a constant } C \neq 0} \cdot e^{-\cos x} = C e^{-\cos x} \; (C \neq 0)$$

$$\Rightarrow \quad \boxed{y = C e^{-\cos x}, \quad C \neq 0 \text{ is constant}}$$

Note $y \equiv 0$ also satisfies ①, So $\boxed{y \equiv 0}$ is also a solution.

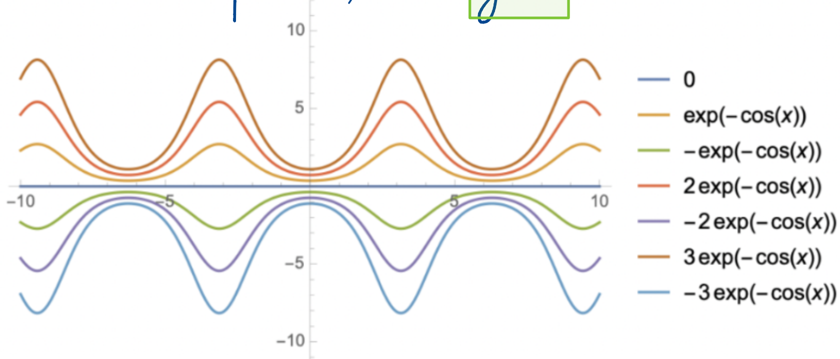

Figure. The solution curves for $\dfrac{dy}{dx} = y \sin x$**.**

General Separable Equations

In general, the first-order differential equation $\frac{dy}{dx} = f(x,y)$ is separable if $f(x,y)$ can be written as the product of a function of x and a function of y:

$$\frac{dy}{dx} = f(x,y) = g(x)k(y)$$

- If $k(y) \neq 0$, then we can write

$$\frac{dy}{k(y)} = g(x)dx$$

- To solve the differential equation we simply integrate both sides:

$$\int \frac{dy}{k(y)} = \int g(x)dx + C$$

- Note we also need to check if $k(y) = 0$ gives us a solution.

Implicit, General, and Singular Solutions

- **General solution:** A solution of a differential equation that contains an "arbitrary constant" C.

  For example, in **Example 1**, $y = Ce^{-\cos x}, C \neq 0$ is a constant is a general solution.

- **Singular solution:** Exceptional solutions cannot be obtained from the general solution.

  In **Example 1**, $y = 0$ is a singular solution.

- **Implicit solution** The equation $K(x,y) = 0$ is commonly called an implicit solution of a differential equation if it is satisfied (on some interval) by some solution $y = y(x)$ of the differential equation.

  For example, in **Example 1**, $\ln|y| = e^{-\cos x} + C$ is an implicit solution

Exercise 2. Find solutions of the differential equation $2\sqrt{x}\frac{dy}{dx} = \sqrt{1-y^2}$.

ANS: Note $1-y^2 \geq 0 \Rightarrow -1 \leq y \leq 1$

If $\sqrt{1-y^2} \neq 0$, $x \neq 0$, we have

$$\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{1}{2}\frac{1}{\sqrt{x}}\,dx$$

$$\Rightarrow \quad \sin^{-1} y = \sqrt{x} + C$$

$$\Rightarrow \quad y(x) = \sin(\sqrt{x} + C)$$

If $\sqrt{1-y^2} = 0$, $y(x) \equiv \pm 1$, which also satisfy the given equation

So the equation has general solution

$$\boxed{y(x) = \sin(\sqrt{x} + C)}$$

and singular solutions

$$\boxed{y(x) \equiv \pm 1}$$

Example 3. Find the particular solution if the initial value problem

separable

$$2y\frac{dy}{dx} = \frac{x}{\sqrt{x^2-16}}, \qquad y(5) = 2.$$

ANS: We have

$$\int 2y\,dy = \int \frac{x}{\sqrt{x^2-16}}\,dx$$

$$\int \frac{x}{\sqrt{x^2-16}}\,dx$$

Let $u = x^2-16$, then $du = 2x\,dx$

$\Rightarrow$ $x\,dx = \frac{1}{2}du$

Thus $\int \frac{x}{\sqrt{x^2-16}}\,dx = \int \frac{\frac{1}{2}du}{\sqrt{u}} = \sqrt{u} + C$

$= \sqrt{x^2-16} + C$

$$\Rightarrow y^2 = \sqrt{x^2-16} + c$$

As $y(5) = 2$,

$$4 = 2^2 = \sqrt{5^2-16} + c = 3 + c$$

$$\Rightarrow c = 1.$$

So

$$y^2 = \sqrt{x^2-16} + 1 \quad \text{(implicit solution)}$$

or

$$\boxed{y = \pm\sqrt{\sqrt{x^2-16} + 1}}$$

Exercise 4 Solve the separable differential equation with the initial condition.

$$11x - 8y\sqrt{x^2+1}\frac{dy}{dx} = 0, \quad y(0) = 2$$

ANS: First separate the variables:

$$11x = 8y\sqrt{x^2+1}\,\frac{dy}{dx}$$

If $y \neq 0$, we have

$$\frac{11x}{\sqrt{x^2+1}}\,dx = 8y\,dy$$

Integrate both sides:

$$\int \frac{11x}{\sqrt{x^2+1}}\,dx = \int 8y\,dy \qquad ①$$

To compute the left hand side, we use u-subs. Let $u = x^2+1$, then $du = 2x\,dx$.

Thus $x\,dx = \frac{1}{2}du$. Then

$$\int \frac{11x}{\sqrt{x^2+1}}\,dx = 11\int \frac{\frac{1}{2}du}{\sqrt{u}} = \frac{11}{2}\int u^{-\frac{1}{2}}\,du = \frac{11}{2}\cdot\frac{1}{1-\frac{1}{2}}u^{\frac{1}{2}} + C_1$$

$$= 11\sqrt{x^2+1} + C_1$$

Thus ① becomes

$$11\sqrt{x^2+1} + C_1 = 4y^2$$

$$\Rightarrow y^2 = \frac{11}{4}\sqrt{x^2+1} + \cancel{\frac{C_1}{4}}$$ is also a constant. call it C.

$$\Rightarrow y^2 = \frac{11}{4}\sqrt{x^2+1} + C.$$

$$\Rightarrow y = \pm\sqrt{\frac{11}{4}\sqrt{x^2+1} + C}$$

Thus $y(x)$ is either ≥ 0 or ≤ 0.

As $y(0)=2 \geq 0$, we have to take the "+" sign.

We have

$$y(0) = 2 = \sqrt{\frac{11}{4}\sqrt{0+1} + C}$$

$$\Rightarrow 4 = \frac{11}{4} + C$$

$$\Rightarrow C = 4 - \frac{11}{4} = 1.25$$

Thus

$$y(x) = \sqrt{\frac{11}{4}\sqrt{x^2+1} + 1.25}$$

Exercise 5 Using separation of variables, solve the differential equation,

$$(6+x^6)\frac{dy}{dx}=\frac{x^5}{y}$$

ANS: We have

$$y\,dy = \frac{x^5}{6+x^6}\,dx$$

since $x^5 dx = \frac{1}{6} dx^6$

$\Rightarrow x^5 dx = \frac{1}{6} d(6+x^6)$

$$\Rightarrow \int y\,dy = \int \frac{x^5}{6+x^6}\,dx = \int \frac{\frac{1}{6}d(6+x^6)}{6+x^6}$$

$$\Rightarrow \frac{y^2}{2} = \frac{1}{6}\ln|6+x^6| + C = \frac{1}{6}\ln(6+x^6)+C$$

since $6+x^6$ is always positive.

Thus

$$y^2 = \frac{1}{3}\ln(6+x^6)+C$$